EVOLUTION OF INTEGRATION
From Antiguity to Riemann

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Contents

INTRODUCTION

The aim of this presentation is to trace the development of the concept of Integration from the Greek period (about 400 B.C.) to the modern times. The presentation is divided into parts :

1. INTEGRATION IN ANTIQUITY (400 B.C. - 200 B.C.)

2. INTEGRATION TILL 18TH CENTURY BEGINNING

3. CONTRIBUTIONS OF CAUCHY (1789 - 1857)

4. CONTRIBUTIONS OF RIEMANN (1826 - 1866)

5. INTEGRATION AFTER RIEMANN

We perceive integration as a mathematical tool related to the physical measurement of lengths/areas/volumes etc.

1 INTEGRATION IN ANTIQUITY

Contributors of this period were:

Antiphon (c. 430 B.C.)

Euclid (c. 300 B.C.)

Archimedes (c. 287 - 212 B.C.)

Their main contributions were:

Calculation of Areas and Volumes.

Their main Technique: Principle of Exhaustion

This provided the Greek mathematicians with a method of proving, in an exact way, results which were already known in some way or other. We give a sample of their work.

Principle of Exhaustion

Given two unequal magnitudes, from the greater of which is subtracted a magnitude larger than its half, and from the remainder a magnitude greater than its half removed, then after a finite number of such operations a quantity is reached which has magnitude less than that of the smaller of the two original magnitudes.

As an application of this, let us prove the following:

Theorem (Proposition (No. 2, Book XII of Elements)):

Areas of circles are in the ratio of squares of their diameter.

For the proof of this theorem, we need the following:

Lemma 1 (Proposition (No. 1, Book XII of Elements)):

Similar polygons inscribed in circles are to one another as the square of their diameters.

Lemma 2: *Given any circle C and any real number $\epsilon > 0$, there exists a polygon P inside C such that*

$$a(C) - a(P) < \epsilon.$$

Proof: Let $a(S)$ denote the area of a region S. On Circle (C), inscribe the square ADBC. Then

$$a(ACDB) = \frac{1}{2}a(PQRS) > \frac{1}{2}a(C).$$

Thus,

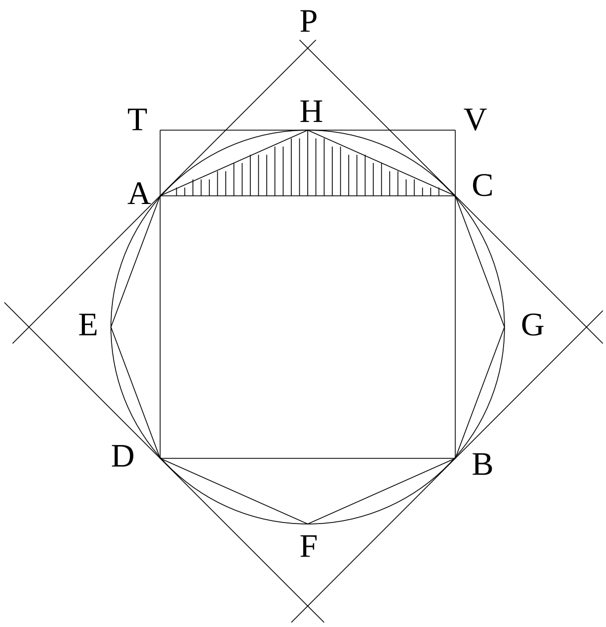
$$a(C) - a(ACBD) := a(2) < \frac{1}{2}a(C).$$

Next, construct the regular octagon AEDFBGCH. Then

$$a(\Delta AHC) = \frac{1}{2}a(ACVT) > \frac{1}{2}a(sector AHC).$$

By similar arguments, we see that the octagon AEDFBGCH includes not only the square ADBC but also includes an area which is more than half the area between circle (C) and the square ACBD. Thus

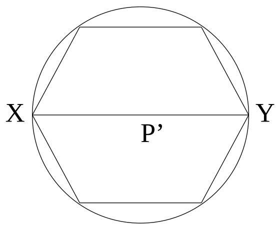
$$a(C) - a(AEDFBGCH) < \frac{1}{2}a(2) < \frac{1}{4}a(C).$$



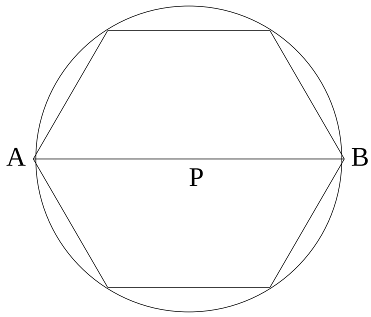
If we continue this process, by the Principle of Exhaustion, after finite number of steps, we will get a regular polygon P inside circle (C) such that a(C) - a(P) is smaller than ϵ.
∎

Now let us prove the theorem:

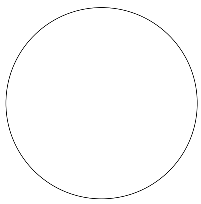
Proof of Theorem: Let circles (1) and (2) have diameters XY and AB, respectively. Let a(F) denote the area of a region F.



(1)



(2)



(3)

Suppose $XY^2 : AB^2 \neq a(1) : a(2)$. Let there exists a circle (3) such that

$$XY^2 : AB^2 = a(1) : a(3). \qquad (*)$$

(The existence of the fourth proportional as an area is assumed by Euclid). Then either a(3) is larger than a(2) or is smaller than a(2). Without loss of generality, let

$$a(3) < a(2).$$

Now by Lemma 2, we can find a regular polygon P inside circle (2) such that a(2) - a(P) is smaller than any given magnitude, say a(2) - a(3). Thus,

$$a(2) - a(P) < a(2) - a(3),$$

i.e.,

$$a(P) > a(3). \qquad (**)$$

∎

Now inscribe inside circle (1) a polygon P' similar to P. Then, by Lemma 1 and (*),

$$XY^2 : AB^2 = a(P') : a(P) = a(1) : a(3).$$

Thus,

$$a(P') : a(1) = a(P) : a(3).$$

Since $a(P') < a(1)$, as P' is inscribed in circle (1), we get $a(P) < a(3)$, a contradiction to (**). Hence the theorem. ∎

An immediate consequence of this theorem is:

Corollary :

Let π denote the area of the unit circle. Then, the area of a circle of radius r is πr^2.

Note :

Using similar arguments, Archimedes computed many other areas and volumes. He also found approximations for the number π by inscribing and circumscribing regular polygons, e.g., he obtained the relation.

$$\frac{223}{71} < \pi < \frac{22}{7}.$$

Another form of Principle of Exhaustion :

Let $a < b$. Let $0 < \alpha < 1$ be such that when we subtract from b a magnitude more than $\frac{1}{2}$ of b, then the remainder is αb. At the next stage the remainder will be $\alpha^2 b$. After n steps what remains is $\alpha^n b$. By Principle of Exhaustion, there exiss a n such that $\alpha^n b < a$, i.e., $\alpha^n < \frac{a}{b}$. Thus, given $\epsilon = \frac{a}{b}$, there exists a n such that $\alpha^n < \epsilon$. In the modern language we say

$$\lim_{n\to\infty} \alpha^n = 0.$$

This is also known as the **Archimedian property** of the real numbers. (Greeks did not have the concept of a real numbers, they knew numbers through magnitudes only. Real numbers were defined only in 1872 by Richard Dedikind and George Cantor).

Remarks on the Greek methods :

(1) They were rigorous and were derived from well stated axioms.
(2) They successfully avoided the concept of limits.
(3) Concepts like Area, Volumes were not defined, but only methods of computing each individually was given.

2 INTEGRATION TILL THE BEGINNING OF 18th CENTURY

For several centuries, till the 17th century, mathematics had algebraic character and not much advancement took place in the field of analysis. As far as the integration is concerned, the main contributors were :

1. Simon Stevin (1548-1620)
2. J. Kepler (1571-1640)
3. B. Cavalieri (1598-1647)
4. Pierre de Fermat (1601-1665)
5. Gregory St. Vincent (1584-1667)
6. John Wallis (1616-1703)
7. Christian Huygens (1629-1695)
8. Pietro Mengoli (1626-1686)
9. Issac Newton (1642-1727)
10. G.W. Leibniz (1646-1716)

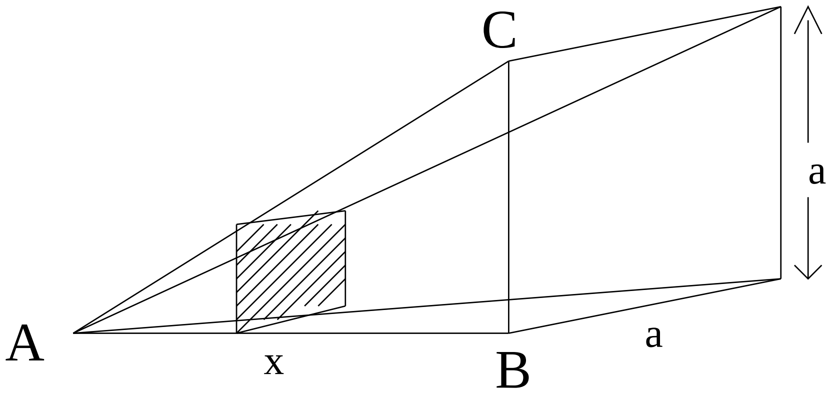
John Kepler, the creater of modern astronomy gave the following:

1. Calculation of volume of wine Casks ("Nova stereometria doliorum vinariorum").
2. Calculation of areas to support the Kepler's second law. "The focal radius joining a planet to the sun sweeps out equal areas in equal times."

Bonvaventura Cavalier, a student of Galileo gave the following :

Cavalieri's Principle
If two solids have equal altitudes, and if sections made by planes parallel to the bases and at equal distances from them are always in a given ratio, then the volumes of the solids are also in this ratio.
(Note the limiting procedure is hidden in this principle.)

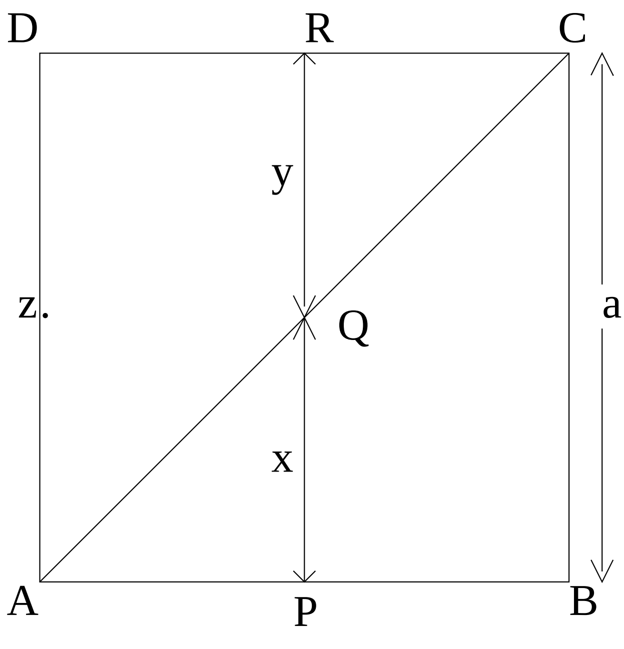
In addition he also devised a method of calculating the volume of a single solid in terms of its cross-sections. For example consider a pyramid with vertex at A and its base being a square on BC.



Then its cross-section at a distance x from the vertex A has area x^2, so

$$Volume(P) = \sum_A^B x^2$$

The same sum is also the area under the parabola $y = x^2$. To compute $\sum_A^B x^2$, Cavalier used the method of sums of squares of lines in a triangle, as follows:



Consider the square ABCD with edge length a divided into two triangles by its diagonal AC. Let x and y denote the lengths of typical sections PQ and QR of these congruent triangles, $x + y = a$. Then

$$\begin{aligned}\sum_A^B a^2 = \sum_A^B (x+y)^2 &= \sum_A^B x^2 + \sum_A^B y^2 + 2\sum_A^B xy \\ &= 2\sum_A^B x^2 + 2\sum_A^B xy \quad (\textit{By symmetry}) \\ &= 2\sum_A^B x^2 + 2\sum_A^B (\frac{a^2}{4} - z^2).\end{aligned}$$

Since $x = \frac{a}{2} - z$ and $y = \frac{a}{2} + z$. Thus,

$$\sum_A^B a^2 = 2\sum_A^B x^2 + \frac{1}{2}\sum_A^B a^2 - 2\sum_A^B z^2.$$

Hence

$$\frac{1}{2}\sum_A^B a^2 = 2\sum_A^B x^2 - 2\sum_A^B z^2.$$

But $\sum_A^B z^2$, by geometry, is $\frac{1}{4}\sum_A^B x^2$. Thus

$$\frac{1}{2}\sum_A^B a^2 = 2\sum_A^B x^2 - \frac{1}{2}\sum_A^B x^2 = \frac{3}{2}\sum_A^B x^2.$$

Hence

$$\sum_A^B x^2 = \frac{2}{3}\frac{1}{2}\sum_A^B a^2 = \frac{1}{3}\sum_A^B a^2 = \frac{a^3}{3},$$

since $\sum_A^B a^2$ is the volume of a cube of side a. Hence $\int_0^a x^2 = \frac{a^3}{3}$.

Pierre de Fermat, the well known French mathematician, gave more or less rigorous proofs of the general formula:

$$\int_0^a x^k dx = \frac{a^{k+1}}{k+1}$$

(Of course he assumed the fact that

$$\lim_{n\to\infty} \frac{1^k + 2^k + \cdots + n^k}{n^{k+1}} = \frac{1}{k+1}).$$

John Wallis gave integration of fractional powers of x, i.e., $\int_0^a x^k dx$, when k a fraction.

Gregory St. Vincent showed that the arc below the curve $\frac{1}{x}$ is a logarithm.

Contributions of Newton and Leibniz

For both of them, integration is to find the area between the graph of a function $y = f(x)$ and the x-axis. Let us fix a point $x = a$ and denote $Z = F(x)$, the area under $f(x)$ between a and x. Both claimed that $f(x)$ is the derivative of $F(x)$.

Newton's justification

Imagine the segment BD at x moves over the area under consideration by Δx. Then the area increases by $\Delta Z = F(x + \Delta x) - F(x)$ which is approximately $f(x)\Delta x$, and 'in the limit' we get

$$\frac{dz}{dx} = f(x).$$

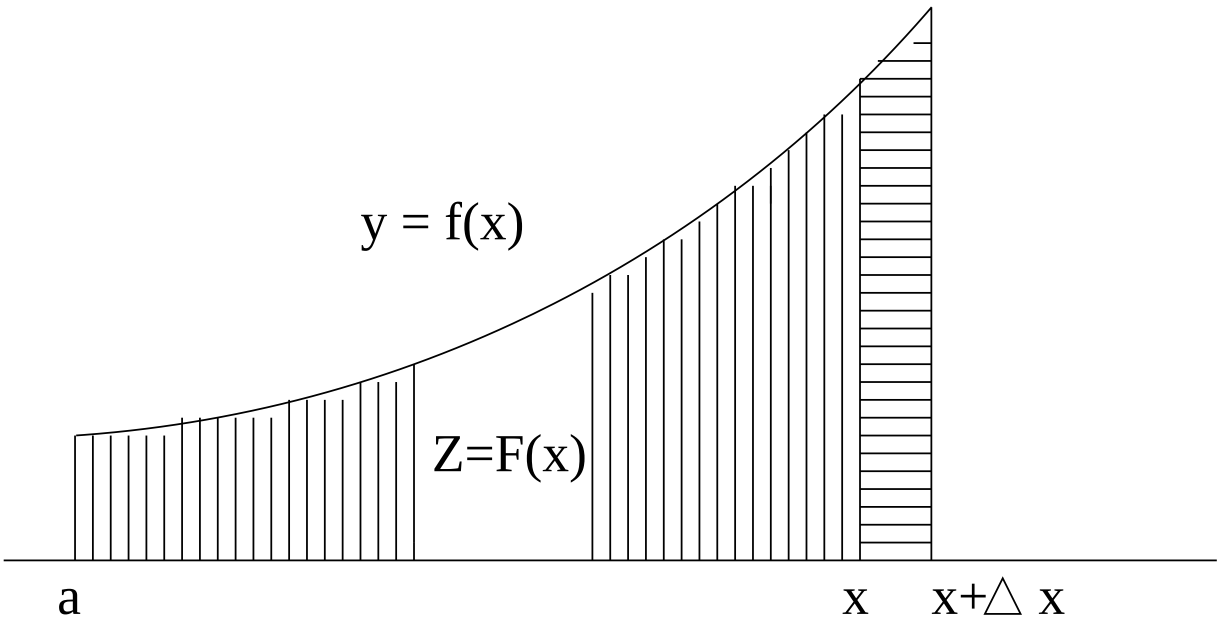


Figure 1: Newton Method

Leibniz's argument

He imagines the area as a sum of small rectangles:

$$Z_n := f(x_1)\Delta x_1 + \cdots + f(x_n)\Delta x_n.$$

This implies

$$Z_n - Z_{n-1} = f(x_n)\Delta x_n.$$

As $\Delta x_i \to 0$, we get $\frac{dz}{dx} = f(x)$.

The contributions of Newton and Leibniz can be summarized as follows:
1. The recognition of the inverse relationship between integration and differentiation.
2. The recognition of the two types of Calculii as new mathematical subjects and not merely a set of useful tricks for solving geometric problems.
3. Systematic derivations of the rules of these subjects and realizing the logical difficulties involved in doing so.

The symbol $\int$ for integration is due to Leibniz (1686) and the term 'integral' was given by Joh. Bernoulli and was published by his brother Jac. Bernoulli in 1690.

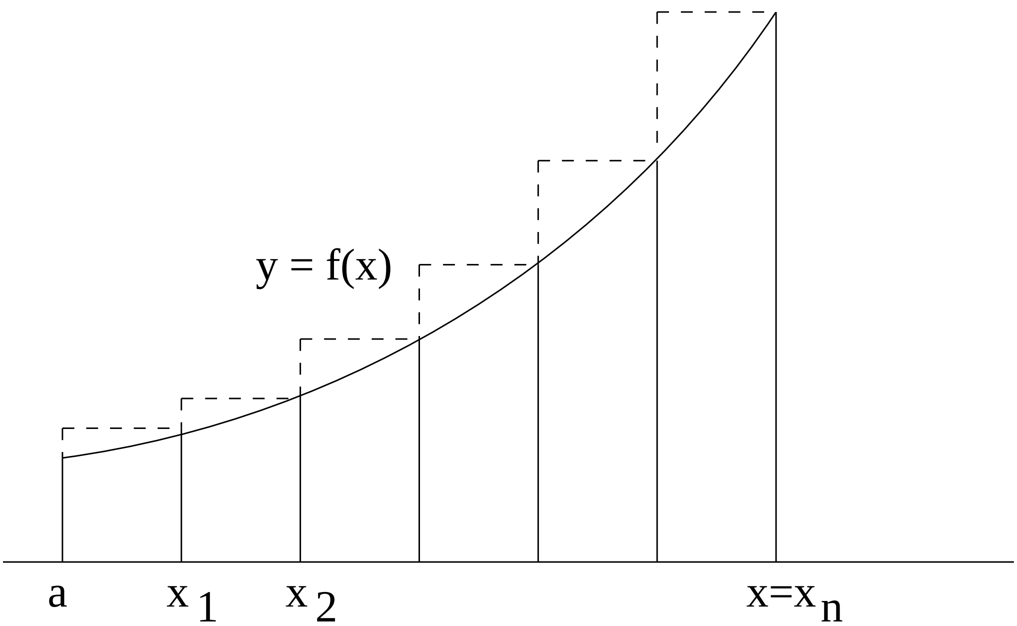


Figure 2: Leibniz Method

Role played by the concept of a function

Though the notion of 'variables' have been used by mathematicians over centuries, the term 'function' was first introduced by Leibniz. He treated various geometric quantities associated with a curve, like tangents, normals, etc., being 'functions' of the curve. The term 'function' was taken over by the Bernoulli's and gradually the function concept lost its geometric character. Johann Bernoulli later defined it as follows: "A function of a variable quantity is a quantity composed in some way or other of this variable quantity and constants." Euler in 1748 gave the following definition of a function: "Every analytic expression in which apart from a variable quantity all quantities that compose this expression are constants, is a function." For example $f(x) = x^2 sin55x + 25x + 10, 0 \leq x \leq 3$ was a function whereas

$$g(x) = \begin{cases} x+2, & 1 \leq x \leq 2 \\ x^2, & 2 < x \leq 3 \end{cases}$$

was not a function.
The discussions between d'Alembert (1747), Euler (1748) and Daniel Bernoulli (1753) on the solution of the problem of vibrating string started a debate on the concept of function.
The 'vibrating string problem' is the following: consider a string of length l put on the x-axis with ends tied at $x = 0$ and $x = l$. If the string is plucked at an initial position $f(a)$ and allowed to vibrate, its motion is governed by

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}, \qquad (1)$$

where $y(x,t)$ denotes the position of a point x at time t. The conditions for motion are:

$$y(0,t) = y(l,t) = 0 \; \forall \; t \; (boundary \; conditions),$$

$$(initial \; condition) \left\{ \begin{array}{lll} y(x,0) & = & f(x) \\ \frac{\partial y}{\partial t}(x,0) & = & 0. \end{array} \right\} \; for \; every \; x.$$

The solutions proposed for this problem were as follows :

d'Alembert's solution:
He said that (1) is satisfied by any function of the form:

$$y(x,t) = \frac{1}{2}[\phi(x+at) - \psi(x-at)]$$

where ϕ and ψ are "arbitrary functions" of a single variable. With the boundary conditions and the initial conditions as above, the solution is

$$y(x,t) = \frac{1}{2}[f(x+at) - f(x-at)] \tag{2}$$

Euler's solution is essentially the same as d'Alembert's. However, Euler said that ϕ can not be any 'arbitrary' function, but should be an 'analytic' function. So, he suggested that solution of Alembert will work with different 'analytic' functions appearing in (2).

D. Bernoulli gave a totally different solution of (1):

$$y(x,t) = \sum a_n sin\frac{n\pi x}{l} cos\frac{n\pi at}{l},$$

and if it has to satisfy boundary conditions also, then

$$f(x) = y(x,0) = \sum a_n \frac{sin n\pi x}{l}. \tag{3}$$

Bernoulli believed that all functions f can be expressed as in (3).

Euler's objections to Bernoulli's solution:

1. In (3) right hand side is a periodic function while f need not be so.
2. In (3) right hand side is an 'analytic' function while f need not be so, i.e., f being the initial position of the string, may not have an analytic expression for it.

This controversy between Euler, Alembert and Bernoulli continued for a decade. As a result, the following questions became apparent.

1. Does every function has to be given by a single 'analytic expression' or an 'equation'?
2. Does every function has to have a graph ?
3. Does every graph represent a function ?

No answers to the above questions were provided. Mathematicians started accepting that a function can be given by different analytic expressions in its domain. However, it was still believed that a function should have a graph which can be drawn with the free motion of a hand.

The interest in the concept of a function was revived with the work of Josheph Fourier (1807) who, while working on problems of heat conduction, said that "a function is a relation in terms of variables but it can take any values and need not be governed by a common law" (i.e., need not be given by a single formula). He further proposed that any such function defined on $[-\pi, \pi]$ can be written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \; cos \; nx + b_n sin \; nx), \tag{4}$$

where

$$a_n := \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \; cos \; mx \; dx$$

and

$$b_n := \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \; sin \; nx \; dx.$$

It may be pointed that integral was not defined precisely (as we understand today), it was only computed geometrically. When is such a representation (as in (4)) possible, can be known as **Fourier series convergence problem.** Fourier's work raised the following questions:

1. What is a function ?
2. For which functions integral can be defined ?
3. Which functions have representation as given by (4) ?

Answer to the first question was given by Dirichlet in 1822. We shall come back to it a little later. The first rigorous definition of integral was given by Cauchy.

3 CAUCHY'S CONTRIBUTIONS

Cauchy is credited with the founding of the modern age of rigor in mathematics. The concept of limit, continuity and the definition of integral are due to him. (For him continuity was what we call 'uniform continuity' today). Here we talk about his contribution in integration. Before we proceed further, let us have a look at his biographical sketch.

Augustin-Louis Cauchy (1789-1857)

Cauchy was born in Paris in 1789 and in 1805 he entered the Ecole Polytechnique to study engineering. Because of his poor health Lagrange and Laplace advised him to devote himself to mathematics. In 1810 upon completing his training in Civil Engineering from the Ecole des Ponts et Chaussees, he was given commission in the Napoleon's army as a military engineer. He left Paris for Cherbourg on his first assignment. Despite a busy schedule as an engineer, he found time for assisting local authorities in conducting school examinations and doing research. In 1811 he submitted his first work on the theory of polyhedra to the Academie des Sciences. The second part of his work was submitted in 1812. In 1813 he returned to Paris and became a Professor at the Ecole Polytechnique. Later he taught at the Faculte des Sciences and at the College de France. In 1830 after Charles was unseated, Cauchy who had sworn a solemn oath of allegiance to Charles, resigned his professorship and exiled himself to Turin. There he taught Latin and Italian for some years. In 1833 he tutored the grandson of Charles X at Prague. In 1838 he returned to Paris where he served as professor in several religious institutions. In 1848 after the revolution, government did away with oaths of allegiance and Cauchy took over the chair of mathematical astronomy at Faculty de Science at Sorbonne. He produced over 500 papers in diverse branches of mathematics in the last nineteen years of his life and died in 1857. He was the pioneer of rigor in mathematical analysis, created abstract theory of groups and founded the theory of elasticity. He advanced the theory of determinant and contributed basic theorems in ordinary and partial differential equations and complex function theory.

Cauchy's definition of integral (1823)

Let $f : [a, b] \to I\!R$ be a continuous function. Consider any partition $P := \{a = a_0 < a_1 < \cdots < a_n = b\}$ of $[a, b]$. Choose points $\xi_i \in [a_{i-1}, a_i], i = 1, 2, \cdots, n$. Let

$$S(P, f) = \sum_{i=1}^{n} f(\xi_i)(a_i - a_{i-1}).$$

We call $S(P, f)$ a Cauchy sum of f with respect to the partition P. This number depends not only on P, but also on the choice of the points ξ_i for a given partition P. Cauchy showed that $\lim_{||P||\to 0} S(P, f)$ exists, i.e., there existed a number L such that for every

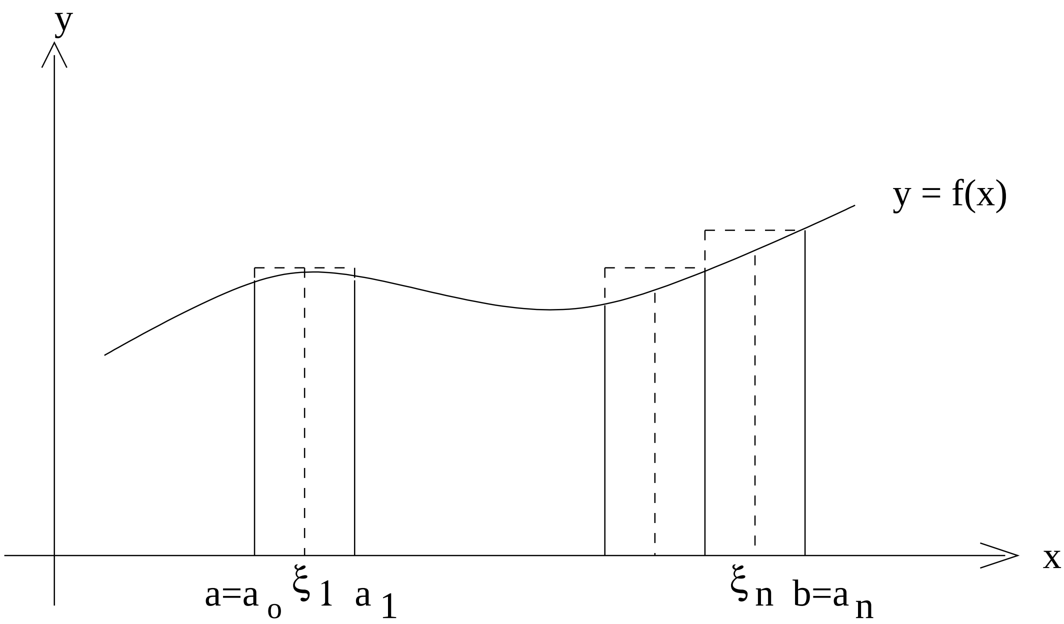


Figure 3: Cauchy Method

$\epsilon > 0$, one can find a $\delta > 0$ such that for every partition P of $[a, b]$ with $||P|| < \delta$ and every Cauchy sum $S(P, f)$, we have

$$|S(P, f) - L| < \epsilon.$$

L is called the **Cauchy integral of f** on $[a, b]$ and is denoted by $\int_a^b f(t)\ dt$. Here, $||P|| = max \mid x_i\ -\ x_{i-1}| \ \{1 \leq i \leq n\}$

Note:

1. For Cauchy, continuity meant 'uniform continuity', which was only defined in 1871 by Heine. That every continuous function on a closed bounded interval is uniformly continuous was proved in 1873.
2. Cauchy also assumed the completeness of real numbers, which were defined only in 1872.
3. He also proved the fundamental theorem of calculus.

Fundamental Theorem of Calculus:
Let $f : [a, b] \rightarrow I\!R$ be a continuous function. Then a function $F : [a, b] \rightarrow I\!R$ satisfies the relation.

$$F(g) - F(x) = \int_x^y f(t)dt, \forall\ a \leq x < y \leq b$$

iff $F^{'}(x) = f(x)\ \forall\ x\ \in\ [a, b]$.

Consequences :

1. A rigorous definition of area below a curve was given.
2. Fundamental theorem of calculus established rigorously that integration and differentiation are inverse of each other and it allowed one to compute integrals. Fourier-coefficient could be defined more rigorously. We give next some applications of Cauchy's integral.

As an application of Cauchy's integral, we show how elementary transcendental functions can be defined analytically.

Applications of Cauchy's integral

1. Definition of the Natural logarithmic function:

For $x\ \in\ I\!R,\ x > 0$ let

$$log(x) = \begin{cases} \int_1^x \frac{1}{t}dt & \text{if } x > 1 \\ 0 & \text{if } x = 1 \\ \int_1^x \frac{1}{t}dt & \text{if } x < 1 \end{cases}$$

$log(x)$ is called the **natural logarithmic function.** It is easy to check using thsi basic results of differential calculus and the Fundamental Theorem of Calculus that the function $log(x)$ has the following properties :

1. $log(1)\ =\ 0$.
2. $log(x)$ is differentiable and is strictly increasing.
3. $log(xy) = log(x) + log(y),\ log(x/y) = log(x) - log(y)$ and $log(x^r) = rlog(x) \forall x > 0, r \in I\!R$.
4. $log(x)$ is concave.

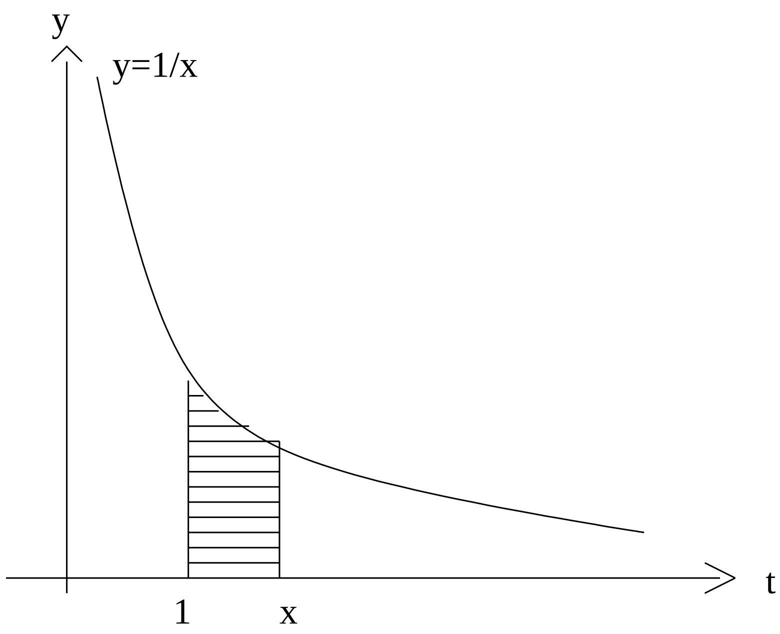


Figure 4: Natural Logarithmic Function

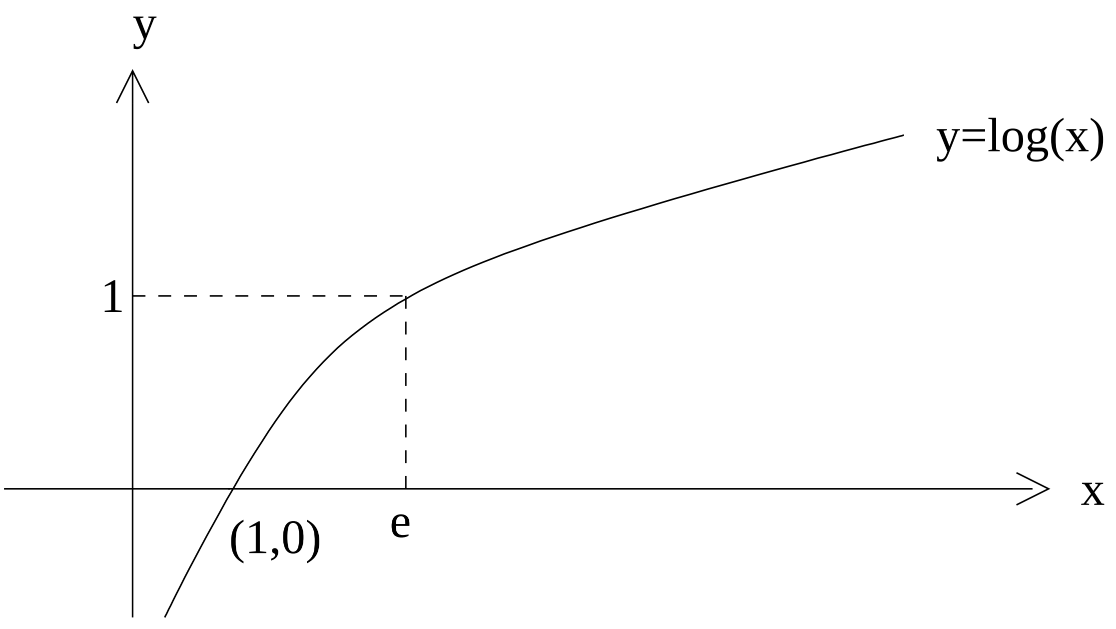


Figure 5: Logarithmic Function

5. $\lim_{x\to\infty} log(x) = +\infty, \lim_{x\to\infty} log(x) = -\infty$.
   Thus, $log : (0, \infty) \to (-\infty, +\infty)$ is a one-one, onto function.

2. Definition of the Exponential function

Since $log : (0, \infty) \to (-\infty, +\infty)$ is a one-to-one function, it has inverse. Let the inverse function called the **exponential function**, be denoted by $\exp : (-\infty, +\infty) \to (0, \infty)$,

$$\exp(x) := \ y \text{ iff } log(y) \ = \ x, \ x \ \in \ I\!R.$$

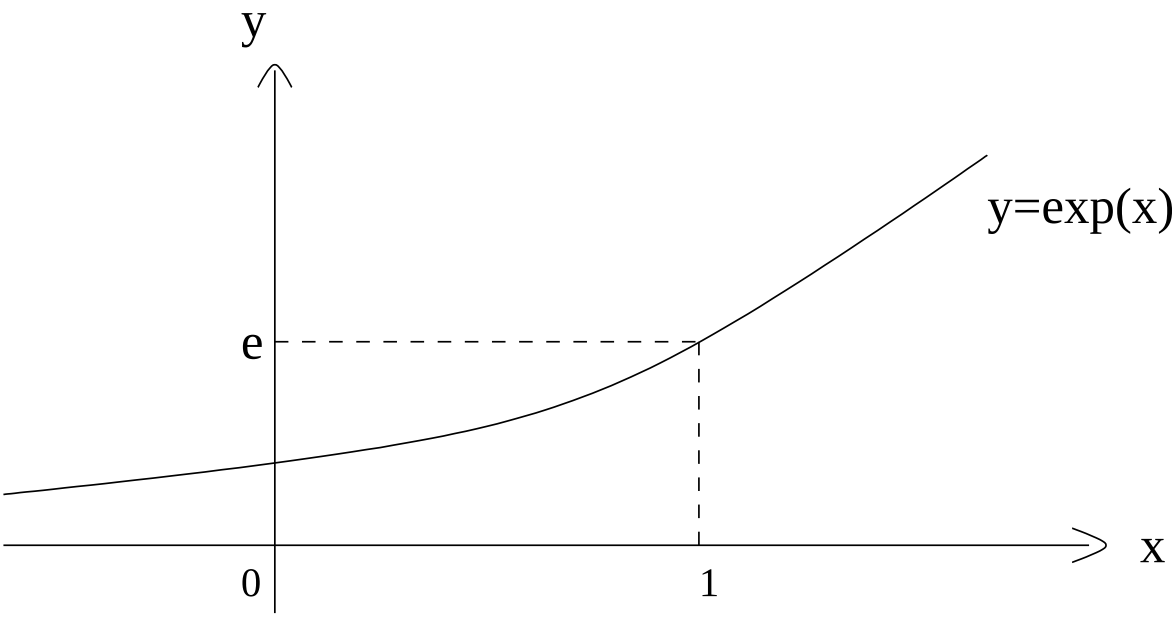


Figure 6: Exponential Function

The exponential function and has the following properties:

1. $\exp(x) > 0 \ \forall \ x \ \in \ I\!R$ and $\exp(0) \ = \ 1$.
2. $\exp(x)$ is differentiable and is a strictly increasing convex function.
3. $\exp(x + y) = \exp(x)\exp(y), \ \exp(x - y) = \exp(x)/\exp(y)$.
4. $\lim_{x\to\infty} \exp(x) = +\infty, \lim_{x\to-\infty} \exp(x) = 0$.

Definition of 'e'-the Euler's Number

Since, $log : (0, \infty) \to (-\infty, +\infty)$ is a one-one onto function, there exists a unique number $x \ \in \ (0, \infty)$ such that $log(x) = 1$. This number is denoted by $\tilde{e}$ and is called the

Euler's number. Thus $log(e) = 1$.

Note: The Euler's number e can also be defined by

$$e := \lim_{n\to\infty}(1 + \frac{1}{n})^n.$$

Let us assume this limit exists and call it $\tilde{e}$. Then, log being differentiable (hence continuous also), we have

$$\begin{aligned} log(\tilde{e}) &= \lim_{n\to\infty} log[(1 + \frac{1}{n})^n] \\ &= \lim_{n\to\infty} \frac{log(1 + \frac{1}{n})}{\frac{1}{n}} \\ &= \frac{d}{dx}(log(x))|_{x=1} \\ &= 1. \end{aligned}$$

Hence $\tilde{e} = e$, since log is one-one.

4. Definition of the number π

In our calculus books we define trigonometric functions geometrically as a periodic function of period 2π, where π is the area of the unit circle. We define the limit by:

$$\lim_{x\to 0} \frac{sinx}{x} = 1.$$

This limit is in turn used to compute the derivative of the sine function, namely the cosine function. Using this together with the trigonometric identities, one computes the area A of the unit circle, via integration, to be

$$A = 4\int_0^1 \sqrt{1 - x^2}dx = \pi.$$

Thus, we calculate what we started with (which we assumed we know) and feel satisfied. In mathematics this presents a logical fallacy, which should be removed. Integration helps us to do so. We first define the trigonometric functions.

5. Definition of trigonometric functions and the number π

Consider the function

$$arc\ sin : [-1, +1] \to I\!R$$

defined by

$$arc\ sin(x) := \begin{cases} \int_0^x \frac{1}{\sqrt{1-t^2}}dt & \text{for } -1 < x < 1 \\ \lim_{x\to 1, x<1} \int_0^x \frac{1}{\sqrt{1-t^2}}dt & \text{for } x = 1 \\ \lim_{x\to 1, x<1} \int_0^x \frac{1}{\sqrt{1-t^2}}dt & \text{for } x = -1 \end{cases}$$

Clearly, $arc\ sin(x)$ is well defined for $|x| < 1$. Since for $0 \leq t < 1$,

$$0 < \frac{1}{\sqrt{1-t^2}} = \frac{1}{\sqrt{1-t}\sqrt{1+t}} < \frac{1}{\sqrt{1-t}},$$

for $0 < x < 1$, we have

$$\int_0^x \frac{1}{\sqrt{1-t}}dt = 2 - 2\sqrt{1-x} \leq 2,$$

thus $g(x) := \int_0^x \frac{1}{\sqrt{1-t^2}}dt \leq 2 \forall x < 1$ and $g(x)$ is increasing as $x \to 1$. Thus, $arc\ sin(1)$ is well-defined. Similarly $arc\ sin(-1)$ is well-defined. It has the following properties:

Properties :

1. $arc\ sin(x)$ is an odd function.
2. Clearly, it is differentiable on $(-1, +1)$, by the fundamental theorem of calculus, and is continuous on $[-1, +1]$, by definition. Its derivative is $\frac{1}{\sqrt{1-t^2}} > 0$ for $t \in (-1, +1)$ and hence it is a strictly increasing function.

Let us define the number $\pi \in I\!R$ by

$$\pi := 2\ arc\ sin(1). \qquad (*)$$

Then,

$$arc\ sin(1) = \frac{\pi}{2} > arc\ sin(0) = 0,$$

and

$$arc\ sin(-1) = -\frac{\pi}{2}.$$

Thus, $arc\ sin : [-1, +1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ is a one-one, onto function.

The inverse function is denoted by sin (called sine) and is given by

$$sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, +1].$$

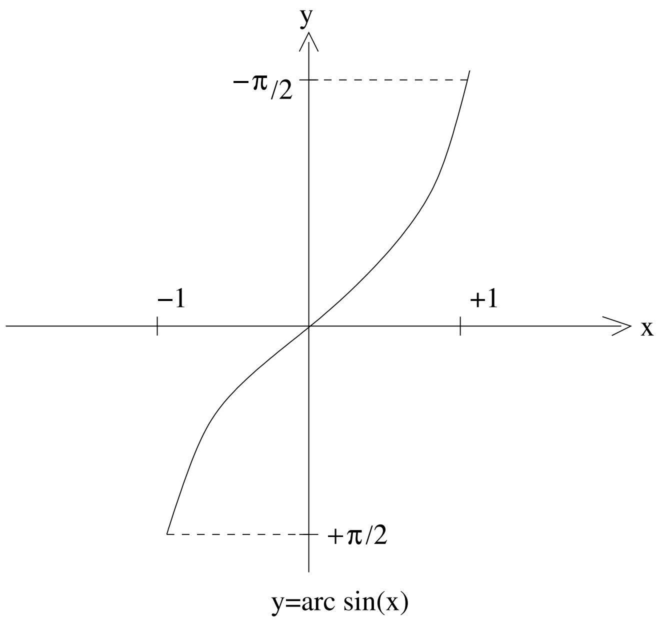
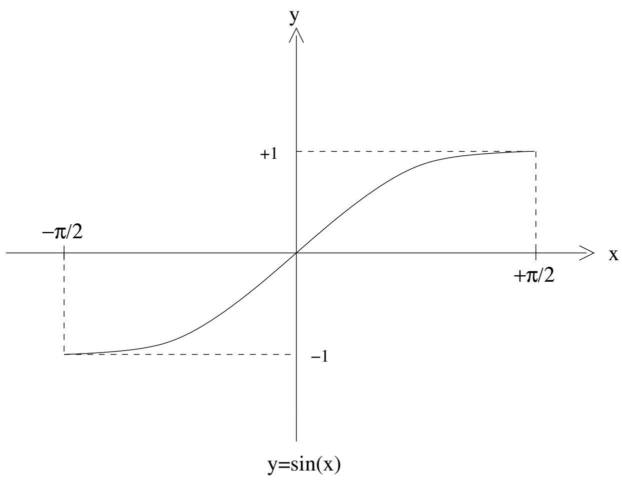


Figure 7: Arc Sine Function



π

Figure 8: Sine Function

It follows by the inverse function theorem that sin is a continuous, monotonically increasing function. Further for $-\pi/2 < x < \pi/2$, it is differentiable and

$$(sin\ x)' = \sqrt{1 - sin^2 x}.$$

Using Lagrange Mean Value Theorem, it is easy to show that $(sin(x))' = 0$ for $x = \pi/2$ and $x = -\pi/2$. Hence

$$(sin\ x)' = \sqrt{1 - sin^2 x}\ for\ every\ x \in [-\pi/2, \pi/2].$$

We extend this function to $[-\pi, \pi]$ as follows :

$$sin(x) := \begin{cases} sin(\pi - x) & \text{if } x \in (\frac{\pi}{2}, \pi] \\ sin(x + \pi) & \text{if } x \in (-\pi, -\frac{\pi}{2}]. \end{cases}$$

Now, extend this function to $I\!R$ periodically with period 2π. It is easy to check that it is a differentiable function and has the usual properties. We can define the trigonometric cosine function to be the derivative of this function. Other trigonometric functions can be defined similarly and the usual trigonometric identities can be proved.

Claim : π, as defined by (*) is the area of the unit circle, as computed via integration:

$$A = 4\int_0^1 \sqrt{1 - x^2}dx = \pi.$$

Exercise : Assuming that the perimeter of a circle is proportional to its radius, show that the perimeter of the unit circle is 2π.

Exercise : Show that $sinx$ has derivative $\sqrt{1 - sin^2 x}$.

Contribution of Dirichlet (1822)

Even though Cauchy's notion of integral gave a mathematical meaning to the Fourier-coefficients a_n, b_n it was still believed that a function can have only finite number of discontinuities (and hence the notion of integral can be extended to such function). Dirichlet, who was analyzing the problem of convergence of Fourier-series, gave example of a function $f : [0, 1] \to I\!R$ defined as follows :

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

This function, called the **indicator function** on the rationals in [0, 1] has infinite number of discontinuities in every interval. Thus, here was an example of a function which was neither defined by a formula, nor its graph could be drawn. Also since it bounds no area, its Fourier-coefficient by the Cauchy's method can not be defined. As a result, the need was felt to treat continuous and discontinuous functions with equal vigour. Also it was believed that the convergence of Fourier-series problem could be solved for a wider class of functions if the notion of integral could be extended from continuous functions to a wider class of functions. This was done by Riemann (1826-1866) in 1854.

4 RIEMANN'S CONTRIBUTIONS:

B. Riemann who was interested in analyzing the 'convergence of Fourier-series' problem realized the need to define the notion of integral for functions which are not necessarily continuous. He extended the definition of integral by realizing that to define $S(P, f)$, it was not necessary to have the continuity of the function, only f bounded is enough. Thus, he defined $S(P, f)$ for a bounded function f, and called f to be **integral** if $\lim_{||p|| \to 0} S(P, f) := L$ existed. A more elegant and geometric definition of Riemann-integral was given by Darboux in 1875 in terms of the upper and lower sums.

Riemann not only proved that the class of functions which are integrable is quite large, he showed that it is much larger than the class of piecewise continuous functions. He gave example of a function $f : I\!R \to I\!R$ which had infinite number of discontinuities in each subinterval of [a, b], yet was integrable in [a, b]. (Using this function Hankel in 1871 constructed a continuous function which was not differentiable at an infinite set of points. In 1872 Karl Weierstrass surprised the mathematicians by producing an example of a function which was continuous every where but was differentiable nowhere.

Example of a function having infinite number of discontinuities in every subinterval and yet being Riemann integrable :
Consider the function $f : [0, 1] \to [0, 1]$ defined by

$$f(x) := \begin{cases} 0 & \text{if } x \text{ is an irrational in } [0,\ 1] \text{ or } x = 0 \\ 1/q & \text{if } x = p/q \text{ is a rational in lowest terms.} \end{cases}$$

It is not very difficult to show that f is continuous at every irrational and discontinuous at every rational in $[0, 1]$. The graph of f is :
Clearly, for any partition P of $[0, 1]$, $L(P, f) = 0$, because every subinterval will have at least one irrational. Next, let $\epsilon > 0$ be arbitrary. Consider the set $\{x \in [0, 1] | f(x) \geq \frac{\epsilon}{2}\}$. This is a finite set. We can cover these points by intervals of total length less than $\frac{\epsilon}{2}$. Let these intervals be $[x_k, y_k], 1 \leq k \leq n$ with $y_k < x_{k+1} \forall k$. Consider the partition

$$P = \{0 = x_0 < x_1 < y_1 < x_2 < \cdots < x_n < y_n \leq x_{n+1} = 1\}.$$

Let $M_k = sup\{f(x) | x_k \leq x \leq y_k\}$, $\tilde{M}_k = sup\{f(x) | y_{k-1} \leq x \leq x_k\}$. Then each $M_k \leq 1$ and $\tilde{M}_k \leq \epsilon/2$. Further

$$U(P, f) \;\; = \;\; \sum_{k=1}^{n} M_k(y_k - x_k) + \sum_{k=1}^{n+1} \tilde{M}_k(x_k - y_{k-1})$$

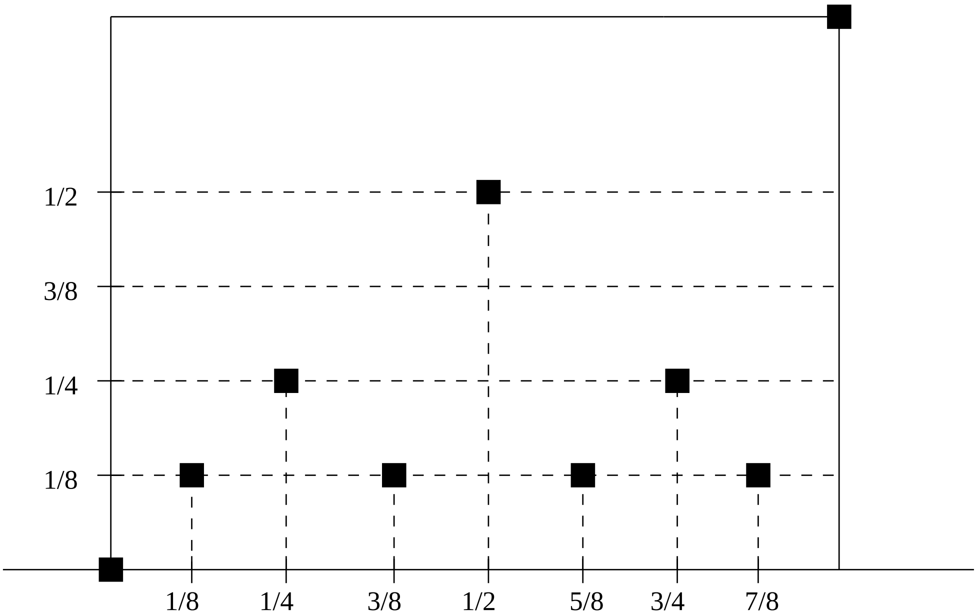


Figure 9: Example of Riemann Function having infinite number of discontinuities

$$\begin{aligned} &\leq \sum_{k=1}^{n}(y_k - x_k) + \frac{\epsilon}{2}\sum_{k=1}^{n+1}(x_k - y_{k-1}) \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence,

$$U(P, f) - L(P, f) \leq \epsilon.$$

Thus, f is Riemann integrable.

One can in fact characterize the class of Riemann-integrable functions. A bounded function $f : [a, b] \to I\!R$ is Riemann-integrable iff f is continuous "almost everywhere".

5 Beyond Riemann-integral

Though Riemann integral gave hope to attack the convergence of Fourier-series problem, the difficulties became obvious.

Drawbacks of the Riemann-integral

(i) The first and the foremost drawback is that the Fundamental Theorem of Calculus is no longer valid for Riemann-integrable functions. For details see Rana[1].

(ii) The second main drawback of Riemann integral is its behavior with respect to limits. Let $\{f_n\}_{n\geq 1}$ be a sequence of Riemann-integrable functions such that $f_n(x) \to f(x)\ \forall x \in [a,b]$, one cannot say that $\int_a^b f_n dx \to \int_a^b f dx$. One has to put strong conditions (like $f_n \to f$ uniformly) to ensure that $\lim_{n\to\infty}(\int_a^b f_n\ dx) = \int_a^b(\lim_{n\to\infty} f_n)\ dx$.

(iii) Consider the space $C[a,b]$, the space of all continuous functions on [a, b]. For $f, g \in C[a,b]$, let

$$d(f,g) = \int_a^b |f(x) - g(x)|\ dx,$$

called the distance between f and g.

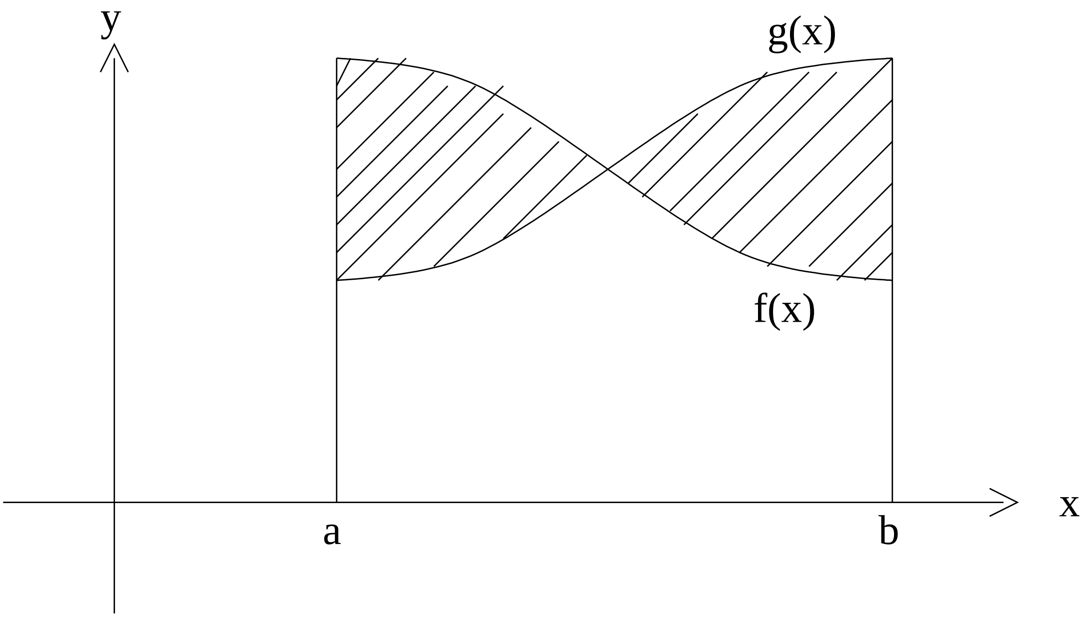


Figure 10: Drawback of Riemann Function

$d(f,g)$ has the following properties:

(a) $d(f,g) \geq 0$, iff $f = g$.

(b) $d(f,g) = d(g,f) \forall f, g \in C[a,b]$.

(c) $d(f,g) \leq d(f,h) + d(h,g), \forall f, g, h \in C[a,b]$.

We know that every $f \in C[a,b]$, is Riemann-integrable. The question arises: is every Cauchy sequence in $C[a,b]$ convergent ? In some sense $\Re[a,b]$ is not complete under the (pseudo) metric d. One asks the following question: What is its completion ?

Above questions along with the need to analyze the 'convergence of Fourier-series' problem motivated the future development of the notion of integral. Efforts of Camille Jordan, Emile Borel, Rene Baire and others culminated in the works of Henri Lebesgue who in 1902 announced a generalization of Riemann-integral, which is now known as **Lebesgue Integral**. For details, see Rana[2]. There is another generalization of Riemann integral called **Henstock Integral**.For details see Bartle[1].

6 References

[1] Bartle, Robert G. *A Modern Theory of Integration,* Graduate Studies in Mathemaics, Volume 32, American Mathematicsl Society,Providence, RI, 2001.

[2] Rana, Inder K. *From Numbers to Analysis*, World Scientific Press, Singapore, 1998.

[3] Rana, Inder K. *An Introduction to Measure and Integration* Graduate Studies in Mathemaics, Volume 45, American Mathematicsl Society,Providence, RI.